

The Philosophy of Mathematics after Foundationalism

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1 Introduction

Philosophy of maths has come to mean something quite specific; namely logic, set theory and the foundations of maths. I'll talk briefly about how this came to be, but mostly I want to talk about other views of the philosophy of mathematics. It seems to me that there are two purposes for the philosophy of mathematics – understanding the meaning of mathematics is the first, and understanding the practice of mathematics is the second. Traditionally, the philosophy of mathematics has concentrated on the meaning of mathematics more than the practice, but recently the emphasis has changed.

The prevalent view of the philosophy of maths is called *foundationalism* – the basic idea is that the job of philosophy is to provide a secure foundation for maths. Simplistically, if you start from true *axioms* and make valid deductions everything that follows is indubitably true. This is occasionally referred to as the Euclidean view of maths, although interestingly some research suggests that the ancient Greeks used the term axiom to mean a statement which we unfortunately cannot prove, rather than the modern usage – a statement which is so obviously true that it doesn't need proof.

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During the eighteenth and nineteenth centuries it began to become clear that the methods of proof that mathematicians were using sometimes gave incorrect results. Most of you will be familiar with examples in analysis – rearranging series which don't converge absolutely for example. These problems with analysis were solved by the $\epsilon - \delta$ formulation of Cauchy and Weierstrass, but more problems kept on appearing. For example, space filling curves violate our intuition of what a curve should be like. Before the discovery of these, it would likely have been considered as “obviously true” that given any curve in the plane, there is at least one point that it misses. If our intuition about curves can be so wrong, obviously there is a need for a formalisation. Most important was the discovery of set theoretic paradoxes. I imagine everyone is familiar with Russell's paradox. Consider the set of all sets which are not members of themselves. Is this set a member of itself or not? Either way leads to a contradiction. Clearly it was necessary to be more precise about what we mean by a set, and what counts as a definition of a set.

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Anyway, this led to the formalisation of mathematics by various people. The attempt to reduce maths to logic, primarily by Russell and Frege was called *logicism*. As an example of how this is done: the number 1 is a property of a predicate ψ that says if both $\psi(x)$ and $\psi(y)$ then $x = y$. In other words, the “class” of all things x satisfying $\psi(x)$ has one member. Frege's system was rendered useless by the discovery of Russell's paradox. Russell's system was problematic in more complicated ways, and was difficult to work with. Eventually, logicism was pretty much abandoned in favour of set theoretic axiomatisations of mathematics by people like Zermelo, Fraenkel and von Neumann (indeed the most common axiomatisation of set theory used today is called ZF after Zermelo and Fraenkel). Logicism had the benefit that if we imagine logic to be unproblematic, and the rules of logic are “obvious” then reducing maths to logic is a solution to the problem of finding a “foundation”. Part of the reason that Russell's programme was abandoned was that his rules of logic were not unproblematic or obvious.

David Hilbert led the *formalist* movement. The idea of formalism is just to find a *formal system* which is enough to generate all of mathematics, and not to worry about what the symbols and rules of this formal system *mean*. A formal system consists of an *alphabet* of basic symbols, a collection of basic true statements (axioms), and a collection of purely symbolic deduction rules. Hilbert hoped that a system would be found that was both *consistent* and *complete*. Consistent means that you

cannot prove a statement to be both true and false. Complete means that every statement can be proved to be either true or false (a statement which can't be proved either true or false is called *undecidable*). In fact, Hilbert hoped that it would be possible to find an intuitively satisfying proof that the system was consistent and complete. Although the formalist *attitude* has survived to this day, the hope of finding complete and consistent systems, and the hope of finding proofs that systems are either consistent or complete has largely faded. In this sense, formalism has failed as a specifically foundationalist philosophy. This is because it isn't enough to just have a formal system that is enough to generate mathematics without having good reasons for thinking it is complete and consistent. It was Gödel that dealt the decisive blow to formalism as a foundationalist philosophy, and I'll try to explain why quickly.

Gödel showed that no formal system which is powerful enough to generate arithmetic can be complete and consistent, and that no such system can be both consistent and prove its own consistency. These are Gödel's *First* and *Second Incompleteness Theorems*. Would anyone like me to explain briefly how these are proved? [Skip to next paragraph if there is no interest.]

In proving Gödel's theorem we employ two formal systems I'll call L and M . The system L I'll refer to as the *object language* and it's the system *about which* we're proving a theorem. The system M I'll refer to as the *metalanguage* and it's the system we will *use* to prove the theorem. We require that there is a way of translating statements in the object language L into statements in the metalanguage M ; that is we use M to *model* L . The proof boils down to the fact that if the object language L contains arithmetic, then we can construct a purely number theoretical statement P which when translated into the metalanguage M means that "The statement P in L is unprovable in L ". This is not self referential because in L , P is only a number theoretical statement, it is only the translation into M which makes P say that P is unprovable. I'm not going to talk about the details of how to construct the statement P as a purely number theoretical statement. Look up *Gödel numbering* on the internet if you're interested, or ask me about it afterwards. The proof then proceeds as follows. If P were false then there would be a proof of P in L , which would make P true and so L would be inconsistent. So if L is consistent then P must be true, which means it has no proof and therefore L is incomplete. This proves Gödel's first incompleteness theorem, and the proof of the second one is exactly the same. Ask me about it afterwards if you're interested.

Talk about soundness of L and details?

Another problem for formalism as a foundationalist philosophy is the existence of various *independence results*. These are examples of particular statements which are of general mathematical interest which can be shown to be neither provable nor disprovable in a particular formal system. Probably most of you know about the *axiom of choice*, which has been shown to be independent of the other axioms of ZF. Similarly, the *continuum hypothesis* has been shown to be independent of the axioms of ZF with the axiom of choice. These results were proved by Gödel and Cohen. They have a great deal of significance for mathematics as a whole. The axiom of choice is necessary in a great many areas of mathematics. For example, it is essential in proving the Hahn-Banach theorem in functional analysis, Tychonoff's theorem in topology, and the existence of maximal ideals in algebra. However, it has some pretty weird consequences. For example, using the axiom of choice you can prove the existence of non-measurable sets. My favourite example is the Banach-Tarski theorem (occasionally called a paradox). This theorem states there is a dissection of a solid sphere of unit radius in \mathbb{R}^3 into six pieces which can be rearranged using rigid transformations alone to form *two* solid spheres of unit radius in \mathbb{R}^3 . (Incidentally, these six sets have to be non-measurable because rigid transformations preserve measure.) Personally, I don't see this as a paradox but simply as showing the limits of the applicability of our spatial intuition.

Hopefully I've given you an idea of the problems with foundationalism in general, and of formalism as a foundationalist philosophy in particular. I'm going to talk about two alternatives to the foundationalist view of the philosophy of mathematics. The first is *quasi-empiricism*, invented by Imre Lakatos. The second view is to reject the idea of foundations altogether. This view is due to Ludwig Wittgenstein, and more recently Hilary Putnam.

I also want to talk about computer proof, some ideas of William Thurston about the nature of mathematics, and some of my own ideas about the possible future of mathematics – this is the subject of the second seminar.

2 Quasi-Empiricism: Imre Lakatos

Imre Lakatos is primarily known nowadays as a philosopher of science, but he did his PhD and early work on the philosophy of mathematics. He was always a great admirer of Karl Popper, and his philosophy of maths – called *quasi-empiricism* – is clearly influenced by Popper’s ideas on the philosophy of science. So much so that Lakatos’ book on the philosophy of maths is called “Proofs and Refutations”, a clear imitation of Popper’s book on the philosophy of science, called “Conjectures and Refutations”. If you know about Popper’s *falsificationist* view of science, many of Lakatos’ ideas will be very familiar.

The aim of foundationalism is to find obviously true axioms and rules of deduction, an unproblematic foundation upon which to build all of mathematics. For Lakatos though, the axioms are more problematic than the results deduced from them. Something like $1 + 1 = 2$ is so obvious that if we defined arithmetic in some formal system, and it turned out that in that system $1 + 1 = 3$, then we would instantly know that we had defined the formal system wrongly. This is an inversion of the foundationalist view.

If we picture the axioms of mathematics as being “at the top” and the consequences of these actions as being “at the bottom”, then we can characterise the foundationalist view of mathematics by saying that “truth flows downwards” from the obviously true statements at the top to the deductions at the bottom. The quasi-empiricist view is that “falsity flows upwards” from consequences at the bottom to the axioms at the top. We already know that it false that $1 + 1 = 3$ regardless of any axiomatisation, so if we can deduce $1 + 1 = 3$ from some set of axioms, the falsity of this statement implies the falsity of the axioms which produced it.

Already this addresses some of the problems with formalism as a foundational philosophy. The problem with formalism is that there is no reason for choosing any particular axiom set over another. Lakatos’ quasi-empiricism begins to solve this problem by saying that we have a pre-existing intuitive idea about what is or isn’t true in mathematics, and we can use this to influence our choice of formal systems to study.

The similarity between this view and scientific empiricism is reasonably clear. In science, we come up with models of the world, or some aspect of the world, and test them against experience. If the model makes incorrect predictions, we reject it. The same is true of quasi-empiricism except that we test the deductions made from our formal systems against our intuitive idea of what mathematics should be like.

Now I’d like to fill out this sketch of quasi-empiricism a little. To begin with, I want to use the term *informal mathematics* to mean what I have been calling our pre-existing intuitive idea of mathematics. I’ll say something more about this slightly vague concept in a minute. There are two types of *falsifiers* of a formal system. A *logical falsifier* is a proof that some particular statement is both true and false. What makes quasi-empiricism interesting is the idea of a *heuristic falsifier*, which occurs when you have a formal proof that some statement is true, and an informal proof that it is false. Heuristic falsifiers are not conclusive falsifiers, because it could be that the informal theory is wrong and the formal theory is right, but the existence of a heuristic falsifier highlights that there is a problem which needs addressing. If the heuristic falsifier is something like $1 + 1 = 3$ then it is pretty clear that the formal system is at fault, but if the heuristic falsifier is something more complex then there is an interesting problem which needs to be thought about carefully.

That is basically all there is to quasi-empiricism as it was described by Lakatos, and it leaves some interesting open questions which are worth thinking about. The most important question is: what is this “informal mathematics” thing that he keeps talking about? Lakatos says somewhat evasively that this is a subject for historical analysis. My feeling is that informal mathematics can be all sorts of things, and each choice of an informal mathematics upon which to base a quasi-empirical philosophy of mathematics tells us something different and interesting. At the simplest, arithmetic and counting is one sort of informal mathematics that we can all agree on. As another possibility, I suggest that theoretical physics is another source of potential heuristic falsifiers for mathematics. For example, the concept of a manifold is obviously tied up with its applications to physics.

There is one other issue which is worth mentioning. Many would say that the formal system ZF of Zermelo and Fraenkel is good enough for axiomatising most mathematics that most mathematicians

spend most of their time on, so that Lakatos' quasi-empiricism may be correct but is now only of historical interest, or of interest only to researchers in obscure branches of set theory and logic. The point is that the quasi-empirical method can be used to explain the development of the content of mathematics as well as the formal systems underlying it. Imagine that we want to define the concept of a real number. We don't just launch in there and say that real numbers are equivalence classes of Cauchy sequences, or Dedekind cuts, we usually come up with a list of properties we would like real numbers to have (like the existence of limits of monotonically increasing bounded sequences, or least upper bounds of bounded sets). We then try to invent constructions which satisfy these definitions. Our intuitive idea of the continuity of the real number line corresponds to the idea of informal mathematics, the list of properties of real numbers is like a list of the potential heuristic falsifiers, and the definition of a real number is the equivalent of the formal system.

I think that quasi-empiricism captures something essential about the way we as mathematicians actually think. This is not coincidental. Lakatos' PhD thesis was strongly influenced by Polya's books on how mathematicians actually solve problems. You might have heard of, and maybe even read his book "How to solve it". Polya gave a very interesting analysis of how you might come up with a proof of Euler's formula $V - E + F = 2$ (the number of vertices minus the number of edges plus the number of faces is 2 for a polyhedron), using basic concepts of mathematical reasoning like generalisation, specialisation, etc. Lakatos took this analysis much further, he was interested in how theorems and mathematical concepts develop over time. So, he starts off describing how this theorem is true for what we initially think of as a polyhedron. Then, he starts coming up with all sorts of exceptions, and develops the statement of the theorem and the definition of polyhedron. I haven't got time to go through all these developments, but if you allow faces to be non-simply connected you get a different result; if the polyhedron is not homeomorphic to a sphere you get a different result ($V - E + F = 0$ for a polyhedron homeomorphic to a torus); etc. Eventually, his development of the concept of a polyhedron and the statement of the theorem lead him to the modern version of this formula in terms of homology.

Clearly this method for analysing how a mathematical concept develops is very powerful. I recently realised I had been using a similar method when developing a concept in my own research. I'm not going to bore you too much with this, I'll just say that I wanted to define the idea of a Jordan curve having an arbitrary amount of spiralling, but no infinite spirals. I already knew there was a very good and well known definition of infinite spiralling, and I wanted my definition for finite but arbitrarily large spiralling to fit with this definition of infinite spiralling in some limiting sense. It took me three or four passes back and forth thinking about examples where my definitions didn't do quite what I wanted them to do before I settled on my final definition. These examples were heuristic falsifiers.

So, I think that quasi-empiricism is not just of historical interest, it can have a continued use in helping us to understand how mathematicians fix upon definitions and problems to study, and maybe it can help us to improve on this process.

Before I move on to talking about Wittgenstein, I want to mention one last thing about quasi-empiricism. There was a philosophical movement called *Pragmatism* which is now largely but not completely ignored. The basic tenet of this movement was to focus not on trying to find out what the objective "truth" was, but to concentrate on how to think usefully. This gives us an interesting alternative view of lots of things that are usually spoken about in terms of truth and falsity. For example, we could say that Newtonian physics is useful because of the predictions it makes, but special and general relativity are more useful in particular cases. This isn't to say that Newtonian physics has been proved false and replaced by special and general relativity, but that it has been discovered that there are certain situations in which it is more useful to apply special and general relativity. This usefulness is not just the everyday sort of usefulness, but usefulness in a very general sense; usefulness as part of human knowledge as a whole, usefulness as part of our attempt to understand and control nature. I find that pragmatism gives me a new and interesting way of thinking about the philosophy of maths, in particular thinking about deciding what we mean by "informal mathematics" in quasi-empiricism. We can imagine coming up with a theory of the development of mathematics based on its usefulness as part of the general human endeavour of understanding and controlling our environment: this would encompass the use of maths in all areas of human life. Important examples would be arithmetic for counting and finance, basic geometry for agriculture and early engineering, calculus and differential equations for physics, and so on right up to modern inventions like stochastic calculus for studying the stock market.

Now, if I have time, I'll move on to talking about Wittgenstein and Putnam.

3 Rejecting Foundationalism: Ludwig Wittgenstein and Hilary Putnam

Wittgenstein's thoughts on foundationalism are much more extreme than Lakatos'. Lakatos basically accepts the idea that you want to come up with a single, unified axiomatic system in which you can do all of mathematics, but Wittgenstein rejects this. His point was that formal proof actually obscures the meaning of the proof.

4 Computer Proof and the Syntactical Approach

5 William Thurston and the Semantical Approach

6 The Future of Mathematics